LMU-TPW 00-25

Nonabelian noncommutative gauge fields and Seiberg-Witten map

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Abstract

Noncommutative gauge fields (similar to the type that arises in string theory with background B-fields) are constructed for arbitrary nonabelian gauge groups with the help of a map that relates ordinary nonabelian and noncommutative gauge theories (Seiberg-Witten map). As in our previous work we employ Kontsevich's formality and the concept of equivalent star products. As a byproduct we obtain a "Mini Seiberg-Witten map" that explicitly relates ordinary abelian and nonabelian gauge fields. (This paper is based on a talk by P. Schupp at the "Brane New World" conference in Torino; for a more detailed version see [8].)

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1 Introduction

The topic of this talk is the type of noncommutative gauge theory that arises in string theory [1]. Since it is not yet clear what exactly the relevant ingredients of the final form of the theory will be we shall not use an axiomatic approach but will instead try to motivate everything from physics as we go along. We choose to work in the framework of deformation quantization because it allows a mathematically rigorous formulation and solution of the questions that interest us here.[1]

Let us start by briefly recall how star products and noncommutative gauge theory arises in string theory: Consider an open string σ-model with background term

$$S_B = \frac{1}{2i} \int_D B_{ij} \, \epsilon^{ab} \partial_a x^i \partial_b x^j, \tag{1}$$

where the integral is over the string world-sheet (disk) and B is constant, nondegenerate and $dB = 0$. The correlation functions on the boundary of the disc in the decoupling limit ($g \to 0$, $\alpha' \to 0$) are

$$\langle f_1(x(\tau_1)) \cdot \ldots \cdot f_n(x(\tau_n)) \rangle_B = \int dx \, f_1 \star \ldots \star f_n, \quad (\tau_1 < \ldots < \tau_n) \tag{2}$$

with the Weyl-Moyal star product

$$(f \star g)(x) = e^{\frac{i\hbar}{2}\theta^{ij}\partial_i \partial'_j} f(x) g(x') \Big|_{x' \to x}, \tag{3}$$

which is the deformation quantization of the Poisson structure $\theta = B^{-1}$. More generally a star product is an associative, $[[\hbar]]$-bilinear product

$$f \star g = fg + \sum_{n=1}^{\infty} (i\hbar)^n \underbrace{B_n(f,g)}_{\text{bilinear}}, \tag{4}$$

which is the deformation of a Poisson structure θ:

$$[f \stackrel{\star}{,} g] = i\hbar\{f \, , \, g\} + \mathcal{O}(\hbar^2), \quad \{f \, , \, g\} = \theta^{ij}(x) \partial_i f \, \partial_j g. \tag{5}$$

[1] Clearly one has to be careful with star products – in particular when studying gauge Morita equivalence of noncommutative tori, where the deformation parameter could be mapped to its inverse [2]. Nevertheless for the construction of the Seiberg-Witten map it appears to be the right language; if there is a solution of the recursion relations that define the map then it should have the form presented here.

Let us now perturb the constant B field by adding a gauge potential $a_i(x)$: $B \to B + da$, $S_B \to S_B + S_a$, with

$$S_a = -i \int_{\partial D} d\tau a_i(x(\tau))\partial_\tau x^i(\tau). \tag{6}$$

Classically we have the naive gauge invariance

$$\delta a_i = \partial_i \lambda, \tag{7}$$

but in the quantum theory this depends on the choice of regularization. For Pauli-Villars (7) remains a symmetry but if one expands $\exp S_a$ and employes a point-splitting regularization then the functional integral is invariant under noncommutative gauge transformations[2]

$$\hat{\delta}\hat{A}_i = \partial_i\hat{\lambda} + i\hat{\lambda} \star \hat{A}_i - i\hat{A}_i \star \hat{\lambda}. \tag{8}$$

Since a senisible quantum theory should be independent of the choice of regularization there should be field redefinitions $\hat{A}(a)$, $\hat{\lambda}(a, \lambda)$ (Seiberg-Witten map) that relate (7) and (8):

$$\hat{A}(a) + \hat{\delta}_{\hat{\lambda}}\hat{A}(a) = \hat{A}(a + \delta_\lambda a). \tag{9}$$

The problems that we shall address are:

- A direct construction of maps $\hat{A}(a)$, $\hat{\lambda}(a, \lambda)$ to all orders in the deformation parameter that satisfy the compatibility condition (9); not term by term via a recursion relation.
- This construction should work for any $\star$-product, i.e., any Poisson structure $\theta(x)$ and it should be covariant under general coordinate transformations. (For this we first need to give a suitably general formulation of noncommutative gauge theory.)
- A further generalization to arbitrary nonabelian gauge groups; not just by simply absorbing a matrix factor into the definition of the algebra.

Sections 2.1 and 3 review the results of [3], [4], [5] in a slightly more general setting. Section 3.2 discusses nonabelian gauge fields [6], [7] in this framework.

In related work a path integral approach [9], [10] and operator methods [12] have been discussed; the role of $\star_n$-operations has been illuminated in [13], [14].

[2]This formula is only valid for the Moyal-Weyl star product.

2 Noncommutative gauge fields

To see how to proceed we note that the extra factor $\exp S_a$ in the correlation function (2) effectively shifts the coordinates[3]

$$x^i \to x^i + \theta^{ij}\hat{A}_j =: \mathcal{D}x^i. \tag{10}$$

More generally, for a function f,

$$f \to f + \mathcal{A}(f) =: \mathcal{D}f. \tag{11}$$

$\mathcal{A}$ plays the role of a generalized gauge potential; it maps a function to a new function that depends on the gauge potential. The shifted coordinates and functions are covariant under noncommutative gauge transformations:

$$\hat{\delta}(\mathcal{D}x^i) = i[\hat{\lambda} \stackrel{\star}{,} \mathcal{D}x^i], \qquad \hat{\delta}(\mathcal{D}f) = i[\hat{\lambda} \stackrel{\star}{,} \mathcal{D}f]. \tag{12}$$

The first expression implies (8) (for θ constant and nondegenerate).

2.1 Covariant coordinates, covariant functions

The covariant coordinates (10) are the background independent operators of [1, 15]; they and the covariant functions (11) can also be introduced more abstractly as follows: Let $\mathcal{A}_x$ be an associative algebra "noncommutative space" with product $\star$.[4] In reference to their commutative analog we shall call the elements of $\mathcal{A}_x$ functions. The gauge transformation of a field $\psi \in$ (left module of) $\mathcal{A}_x$ is

$$\hat{\delta}\psi = i\hat{\lambda} \star \psi, \tag{13}$$

where the gauge parameter $\hat{\lambda} \in \mathcal{A}_x$ is an arbitrary function on the noncommutative space. (More precisely ψ is the coefficient function (or vector) of a section in a projective module over $\mathcal{A}_x$ – we shall come back to this later.) Since the product of a function and a field is not covariant on a noncommutative space,

$$\hat{\delta}(f \star \psi) = f \star \hat{\delta}\psi = f \star (i\hat{\lambda} \star \psi) \neq i\hat{\lambda} \star (f \star \psi), \tag{14}$$

one needs to introduce covariant functions

$$\mathcal{D}f = f + \mathcal{A}(f), \tag{15}$$

[3]Notation: $\mathcal{D}$ should not be confused with a covariant derivative (but it is related).
[4]Until section 3 this does not need to be a star product

where $\mathcal{A} \in \mathrm{Hom}(\mathcal{A}_x, \mathcal{A}_x)$ transforms such that

$$\hat{\delta}(\mathcal{D}f \star \psi) = i\lambda \star (\mathcal{D}f \star \psi), \tag{16}$$

i.e.,

$$\hat{\delta}\mathcal{A}(f) = [i\hat{\lambda} \stackrel{\star}{,} f] + [i\hat{\lambda} \stackrel{\star}{,} \mathcal{A}(f)]. \tag{17}$$

There are other covariant objects:

$$\mathcal{F}(f,g) = [\mathcal{D}f \stackrel{\star}{,} \mathcal{D}g] - \mathcal{D}([f \stackrel{\star}{,} g]), \tag{18}$$

e.g., plays the role of a generalized field strength; it maps two functions to a new function that depends on the gauge potential and transforms covariantly. The generalized field strength is antisymmetric in its arguments, i.e., $\mathcal{F} \in \mathrm{Hom}(\mathcal{A}_x^{\wedge 2}, \mathcal{A}_x)$. For θ constant $\mathcal{A}(x^i) = \theta^{ij}\hat{A}_j$ and

$$\mathcal{F}(x^i, x^j) = i\theta^{ik}\theta^{jl}\hat{F}_{kl}, \qquad \hat{F}_{kl} = \partial_k\hat{A}_l - \partial_l\hat{A}_k - i[\hat{A}_k \stackrel{\star}{,} \hat{A}_l]. \tag{19}$$

There are several reasons, why one needs $\mathcal{A}$ and $\mathcal{D}$ and not just $A^i \equiv \mathcal{A}(x^i)$ (or $\hat{A}_i$, for θ constant): If we perform a general coordinate transformation $x^i \mapsto x^{i'}(x^j)$ and transform A^i (or $\hat{A}_i$) as its index structure suggests, then we will obtain objects that are no longer covariant under noncommutative gauge transformations. The correct transformation is $\mathcal{A}(x^i) \mapsto \mathcal{A}(x^{i'})$. Furthermore we may be interested in covariant versions of scalar fields $\phi(x)$. These are given by the corresponding covariant function $\mathcal{D}(\phi(x))$.

2.2 Abstract formulation of gauge theory on a noncommutative space

Finite projective modules take the place of fiber bundles in the noncommutative realm. This is also the case here, but may not have been apparent since we have been working with component fields as is customary in the physics literature. As we have argued, $\mathcal{A} \in C^1$, $\mathcal{F} \in C^2$ with $C^p = \mathrm{Hom}(\mathcal{A}_x^{\wedge p}, \mathcal{A}_x)$, $C^0 \equiv \mathcal{A}_x$. These p-cochains take the place of forms on a noncommutative space, the wedge product is replaced by the cup product $\wedge : C^p \otimes_{\mathcal{A}_x} C^q \to C^{p+q}$ and the exterior differential is replaced by the standard coboundary operator $\mathbf{d}_\star$ in the Lie-algebra cohomology; $\mathbf{d}_\star^2 = 0$, $\mathbf{d}_\star 1 = 0$. (In the Hochschild cohomology $\mathbf{d}_\star$ can be expressed in terms of the Gerstenhaber bracket, $\mathbf{d}_\star\mathcal{C} = -[\mathcal{C}\,, \star]_{\mathrm{G}}$, and can then be restricted to a map $C^p \to C^{p+1}$ by antisymmetrization.) This calculus uses only the algebraic structure of $\mathcal{A}_x$; it is related to the standard universal calculus and we can obtain other calculi by projection. Consider now a (finite) projective right $\mathcal{A}_x$-module $\mathcal{E}$. We can introduce a connection on $\mathcal{E}$ as a linear map $\nabla : \mathcal{E} \otimes_{\mathcal{A}_x} C^p \to \mathcal{E} \otimes_{\mathcal{A}_x} C^{p+1}$ for $p \in \mathbb{N}_0$ which satisfies the Leibniz rule[5]

$$\nabla(\eta\psi) = (\tilde{\nabla}\eta)\psi + (-)^p \eta\, \tilde{\mathbf{d}}_\star \psi \tag{20}$$

for all $\eta \in \mathcal{E} \otimes_{\mathcal{A}_x} C^p$, $\psi \in C^r$, and where $\tilde{\nabla}\eta = \nabla\eta - (-)^p\eta\, \tilde{\mathbf{d}}_\star 1$,

$$\tilde{\mathbf{d}}_\star(a \wedge \psi) = (\mathbf{d}_\star a) \wedge \psi + (-)^q a \wedge (\tilde{\mathbf{d}}_\star \psi) \tag{21}$$

for all $a \in C^q$, and $\tilde{\mathbf{d}}_\star 1$ is the identity operator on $\mathcal{A}_x$.

Let (η_a) be a generating family for $\mathcal{E}$; any ξ can then be written as $\xi = \sum \eta_a \psi^a$ with $\psi^a \in \mathcal{A}_x$ (with only a finite number of terms different from zero). For a free module the ψ^a are unique, but we shall not assume that. Let the generalized gauge potential be defined by the action of $\tilde{\nabla}$ on the elements of the generating family: $\tilde{\nabla}\eta_a = \eta_b \mathcal{A}^b_a$. In the following we shall suppress indices and simply write $\xi = \eta.\psi$, $\tilde{\nabla}\eta = \eta.\mathcal{A}$ etc. We compute

$$\nabla\xi = \nabla(\eta.\psi) = \eta.(\mathcal{A} \wedge \psi + \tilde{\mathbf{d}}_\star \psi) = \eta.(\mathcal{D} \wedge \psi). \tag{22}$$

Evaluated on a function $f \in \mathcal{A}_x$ this yields a covariant function times the matter field: $(\mathcal{D} \wedge \psi)(f) = (f + \mathcal{A}(f)) \star \psi = (\mathcal{D}f) \star \psi$. Similarily

$$\nabla^2\xi = \eta.(\mathcal{A} \wedge \mathcal{A} + \mathbf{d}_\star \mathcal{A}).\psi = \eta.\mathcal{F}.\psi \tag{23}$$

with the field strength

$$\mathcal{F} = \mathbf{d}_\star \mathcal{A} + \mathcal{A} \wedge \mathcal{A}, \tag{24}$$

which agrees with (18) and satisfies the Bianchi identity

$$\mathbf{d}_\star \mathcal{F} + \mathcal{A} \wedge \mathcal{F} - \mathcal{F} \wedge \mathcal{A} = 0 \tag{25}$$

due to the associativity of $\mathcal{A}_x$. Infinitesimal gauge transformations in the present notation are

$$\delta\mathcal{A} = -i\mathbf{d}_\star \lambda + i\lambda \wedge \mathcal{A} - i\mathcal{A} \wedge \lambda, \tag{26}$$

$$\delta\mathcal{F} = i\lambda \wedge \mathcal{F} - i\mathcal{F} \wedge \lambda. \tag{27}$$

[5] The transformation of matter fields (13) leads to a slight complication here; for fields that transform in the adjoint (by star-commutator) we only need $\tilde{\nabla}$, $\mathbf{d}_\star$.

3 Equivalence of star products and Seiberg-Witten map

Reconsider the correlation function (2) from the point of view of the path-integral representation [16] of Kontsevich's star product. This is not limited to constant Poisson structures, so we can study the result of the perturbation $B \to B + da$ more directly:[6]

$$\begin{aligned} &\langle f(x(0))g(x(1))\delta_x(x(\infty))\rangle_B = (f \star g)(x), \\ &\langle f(x(0))g(x(1))\delta_x(x(\infty))\rangle_{B+da} = (f \star' g)(x). \end{aligned} \tag{28}$$

More generally we can consider Poisson structures θ, θ' (which may be degenerate) in place of the symplectic structures $\omega = B$ and $\omega' = B + da$:

$$\begin{array}{ccc} \text{Poisson structure} & & \text{star product} \\ \theta & \longrightarrow & \star \\ \downarrow & & \\ \theta' & \longrightarrow & \star' \end{array} \tag{29}$$

Interpretation and Strategy: The map from $\star$ to $\star'$ that completes the above diagram is the equivalence map $\mathcal{D} = \mathrm{id} + \mathcal{A}$ defined by the generalized gauge potential $\mathcal{A}$:

$$\mathcal{D}f \star \mathcal{D}g = \mathcal{D}(f \star' g). \tag{30}$$

In the nondegenerate case we can argue for this from Moser's lemma [17]: The symplectic structures ω and $\omega' = \omega + da$ are related by a coordinate transformation ρ^* generated by the vector field $\theta^{ij}a_j\partial_i$:[7]

$$\rho^*\{f, g\}' = \{\rho^* f, \rho^* g\}. \tag{31}$$

It then follows from a theorem due to Kontsevich [18] that the corresponding star products $\star$ and $\star'$ are equivalent. In first order in θ the equivalence map and $\mathcal{A}$ are given by Moser's vector field, i.e., in terms of the ordinary gauge potential a_i. In the next sections we shall construct this equivalence map $\mathcal{D}$ and the generalized noncommutative gauge potential $\mathcal{A}$ explicitly as a function of the ordinary gauge potentials a.

[6]Here we have $g \equiv 0$ from the start and Batalin-Vilkovisky quantization has to be used, while in (2) g was send to zero after computing the correlation function.

[7]More precisely we should also ask that $\omega_t \equiv \omega + t\, da$ is nondegenerate for all $t \in [0, 1]$.

3.1 Abelian case

We shall take an abelian gauge potential, the corresponding field strength and abelian gauge transformations as given data:

$$a = a_i dx^i, \quad f = da, \quad \delta_\lambda a = d\lambda. \tag{32}$$

We will first construct a semiclassical version of the Seiberg-Witten map, where all star commutators are replaced by Poisson brackets. The construction is essentially a formal generalization of Moser's lemma to Poisson manifolds.

3.1.1 Semi-classicaly

Let us introduce the coboundary operator

$$\mathbf{d}_\theta = -[\,\cdot\,,\,\theta]_{\mathrm{S}}, \tag{33}$$

where $[\ ,\ ]_{\mathrm{S}}$ is the Schouten-Nijenhuis bracket of polyvector fields and $\theta = \frac{1}{2}\theta^{ij}\partial_i \wedge \partial_j$ is the Poisson bivector. ($\mathbf{d}_\theta$ is a very useful operator, since it turns a function f into the corresponding Hamiltonian vector field $\mathbf{d}_\theta f = \theta^{ij}(\partial_j f)\partial_i \equiv \{\,\cdot\,,\,f\}$.) Let us define

$$\mathbf{a}_\theta = a_i\,\mathbf{d}_\theta x^i = \theta^{ji} a_i \partial_j \qquad \text{(Moser's vector field)}, \tag{34}$$

$$\mathbf{f}_\theta = \mathbf{d}_\theta \mathbf{a}_\theta = -\frac{1}{2}(\theta \cdot f \cdot \theta)^{ij}\partial_i \wedge \partial_j \tag{35}$$

and introduce a one parameter deformation θ_t of the Poisson structure θ: $t \in [0,1]$, $\theta_0 = \theta$, $\theta_1 =: \theta'$ by the differential equation

$$\partial_t \theta_t = \mathbf{f}_{\theta_t}. \tag{36}$$

(θ_t is Poisson, because $\mathbf{d}_{\theta_t}\mathbf{f}_{\theta_t} = 0$ and if f is not explicitely θ-dependent, we find $\theta_t = \theta - t\,\theta f \theta + t^2\,\theta f \theta f \theta \mp \ldots$, i.e., $\omega_t = \omega + tf$ in the nondegenerate case.) The t-evolution is generated by the vector field $\mathbf{a}_\theta$, because $\mathbf{f}_{\theta_t} = -[\mathbf{a}_{\theta_t}\,,\,\theta_t]_{\mathrm{S}}$. The Poisson structures θ and θ' are related by the flow

$$\rho_a^* = e^{\mathbf{a}_{\theta_t} + \partial_t} e^{-\partial_t}\Big|_{t=0}. \tag{37}$$

Let

$$A_a = \rho_a^* - \mathrm{id}, \qquad \tilde{\lambda} = \sum_{n=0}^{\infty} \frac{(\mathbf{a}_{\theta_t} + \partial_t)^n(\lambda)}{(n+1)!}\Bigg|_{t=0}, \tag{38}$$

then:

$$A_{a+d\lambda} = A_a + \mathbf{d}_\theta \tilde{\lambda} + \{A_a\,,\,\tilde{\lambda}\}. \tag{39}$$

This is the semi-classical version of the Seiberg-Witten condition (9).

3.1.2 Kontsevich formality map

We would now like to quantize the semi-classical solution by lifting the Poisson structures to star products and, indeed, by lifting all polyvector fields to polydifferential operators. This is done by Kontsevich's formality maps [18]:

$$U_0(f) = f, \quad \underbrace{[U_n(\underbrace{\alpha_1, \ldots, \alpha_n}_{\text{polyvectors}})]}_{\text{plolydiff. operator}} (\underbrace{f_1, \ldots, f_m}_{\in C^\infty(\mathcal{M})}) \;\in\; C^\infty(\mathcal{M}), \quad n \in \mathbb{N}, \tag{40}$$

$$\alpha_i = \sum [\alpha_i(x)]^{l_1 \ldots l_{k_i}} \partial_{l_1} \wedge \ldots \wedge \partial_{l_{k_i}}. \tag{41}$$

Some properties of the U_n:

- skew-symmetric, multilinear in all n arguments
- matching condition: $m = 2 - 2n + \sum k_i$
- formality condition (FC) (see [18], [19])

Example: The U_n lift a bivector field θ to the bidifferential operator

$$f \star g = \sum \frac{(i\hbar)^n}{n!} U_n(\theta, \ldots, \theta)(f, g) = fg + \frac{i\hbar}{2}\theta^{ij}\partial_i f \partial_j g + \ldots . \tag{42}$$

If $[\theta\,,\,\theta]_{\mathrm{S}} = 0$ (Poisson) then the formality condition implies that $\star$ is associative. For constant θ (42) is precisely the Moyal-Weyl star product.

3.1.3 Quantum

We lift Moser's vector field $\mathbf{a}_\theta$ via formality to the differential operator

$$\mathbf{a}_\star = \sum \frac{(i\hbar)^n}{n!} U_{n+1}(\mathbf{a}_\theta, \theta, \ldots, \theta), \tag{43}$$

whose transformation properties follow from the formality condition:

$$\delta \mathbf{a}_\theta = \mathbf{d}_\theta \lambda \overset{\text{(FC)}}{\Rightarrow} \delta \mathbf{a}_\star = \frac{1}{i\hbar} \mathbf{d}_\star \bar{\lambda}; \quad \bar{\lambda} \equiv \sum \frac{(i\hbar)^n}{n!} U_{n+1}(\lambda, \theta, \dots, \theta). \tag{44}$$

In analogy to (33) and (34) we have the coboundary operator

$$\mathbf{d}_\star = -[\,\cdot\,, \star]_{\mathrm{G}}, \tag{45}$$

where $[\ ,\ ]_{\mathrm{G}}$ is the Gerstenhaber bracket, and the bidifferential operator

$$\mathbf{f}_\star = \mathbf{d}_\star \mathbf{a}_\star \overset{\text{(FC)}}{=} \sum \frac{(i\hbar)^{(n+1)}}{n!} U_{n+1}(\mathbf{f}_\theta, \theta, \dots, \theta). \tag{46}$$

We lift θ_t to a t-dependent star product

$$g \star_t h = \sum \frac{(i\hbar)^n}{n!} U_n(\theta_t, \dots, \theta_t)(g, h) \overset{\text{(FC)}}{\Rightarrow} \partial_t (g \star_t h) = \mathbf{f}_{\star_t}(g, h). \tag{47}$$

As an operator equation: $\partial_t(\star_t) = \mathbf{f}_{\star_t}$. Since $\mathbf{f}_{\star_t} = -[\mathbf{a}_{\star_t}\,, \star_t]_{\mathrm{G}}$, the t-evolution is generated by the differential operator $\mathbf{a}_\star$. The star products $\star$ and $\star'$ are related by the equivalence map or “quantum flow”

$$\mathcal{D}_a = e^{\mathbf{a}_{\star_t} + \partial_t} e^{-\partial_t} \Big|_{t=0}. \tag{48}$$

Let

$$\mathcal{A}_a = \mathcal{D}_a - \mathrm{id}, \qquad \Lambda(\lambda, a) = \sum \frac{(\mathbf{a}_{\star_t} + \partial_t)^n (\bar{\lambda})}{(n+1)!} \Bigg|_{t=0}, \tag{49}$$

then

$$\mathcal{A}_{a+d\lambda} = \mathcal{A}_a + \frac{1}{i\hbar}(\mathbf{d}_\star \Lambda - \Lambda \star \mathcal{A}_a + \mathcal{A}_a \star \Lambda). \tag{50}$$

In (49) we have thus found the full Seiberg-Witten map for the generalized noncommutative gauge potential $\mathcal{A}$ for an arbitrary Poisson structure $\theta(x)$. In components:

$$\mathcal{A}_a(x^i) = \theta^{ij} a_j + \frac{1}{2} \theta^{kl} a_l (\partial_k(\theta^{ij} a_j) - \theta^{ij} f_{jk}) + \dots, \tag{51}$$

$$\Lambda = \lambda + \frac{1}{2} \theta^{ij} a_j \partial_i \lambda + \frac{1}{6} \theta^{kl} a_l (\partial_k(\theta^{ij} a_j \partial_i \lambda) - \theta^{ij} f_{jk} \partial_i \lambda) + \dots. \tag{52}$$

3.2 Nonabelian case

The given nonabelian data is

$$A_\mu(x) = A_{\mu b}(x) T^b, \quad [T^a, T^b] = i C^{ab}_c T^c,$$
$$\delta_\Lambda A_\mu = \partial_\mu \Lambda + i[\Lambda, A_\mu], \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \tag{53}$$

Our goal is to find maps $\hat{A}(A_\mu)$, $\hat{\Lambda}(A_\mu, \Lambda)$, such that

$$\hat{A}(A_\mu + \delta_\Lambda A_\mu) = \hat{A}(A_\mu) + \hat{\delta}_{\hat{\Lambda}} \hat{A}(A_\mu), \tag{54}$$

with a noncommutative (and nonabelian) gauge transformation $\hat{\delta}$. A naive geralization of the abelian construction immediately runs into several problems: We could try to let $B \to B + F =: B'$ and correspondingly $\Theta \to \Theta'$, but even if $dB = 0$ (Θ Poisson) $dB' \neq 0$ and Θ' will not be Poisson, since $dF = -A \wedge A \neq 0$. Furthermore, the nonabelian analog of Moser's vector field, $\Theta^{\mu\nu} A_\nu D_\mu$, is not a derivation, essentially because A_ν is matrix-valued.

Strategy: We shall introduce a larger space with coordinates x^i, x^j, ... that is the product of noncommutative space-time with coordinates x^μ, x^ν, ... and an internal space with coordinates t^a, t^b, ... The enlarged space shall have a Poisson structure which is the direct sum of external $\Theta^{\mu\nu}$ and internal $\vartheta^{ab} = C^{ab}_c t^c$ Poisson structures

$$\left(\theta^{ij}\right) = \left(\begin{array}{c|c} \Theta^{\mu\nu} & 0 \\ \hline 0 & \vartheta^{ab} \end{array}\right). \tag{55}$$

Then we pick appropriate *abelian* gauge potentials and parameters that are linear in the internal coordinates and quantize everything with the previous method. After quantization of the internal space (and taking an appropriate representation) the symbols t^a become the generators of the Lie algebra with structure constants C^{ab}_c, giving an nonabelian theory on commutative space-time. After quantizing everything we obtain the desired nonabelian noncommutative gauge theory.

3.3 Mini Seiberg-Witten map

As in the abelian case the space-time components of the noncommutative gauge potential are determined via the Seiberg-Witten map by the term of lowest order in Θ:

$$\mathcal{A}_a(x^\mu) = \Theta^{\mu\nu} A_\nu(a) + \mathcal{O}(\Theta^2). \tag{56}$$

From (49) we can compute the nonabelian gauge potential $A_\nu(a)$ and the nonabelian gauge parameter Λ to all orders in C_c^{ab} and in the internal components of the abelian gauge potential a_b:

$$A_\nu(a) = \sum \frac{1}{n!} t^c (M^{n-1})^b_c (a_{\nu b} - (n-1) f_{\nu b}), \tag{57}$$

with $a_{\nu b} t^b = a_\nu$ and the matrix $M^a_b = C^{ac}_b a_c$,

$$\Lambda(\lambda, a) = \sum \frac{1}{(n+1)!} t^a (M^n)^b_a \lambda_b. \tag{58}$$

(In the special gauge of vanishing internal gauge potential $a_b = 0$ this becomes simply $A_\mu(a) = a_\mu$, $\Lambda = \lambda$.)

Acknowledgements

We would like to thank A. Alekseev, P. Aschieri, J. Madore, S. Schraml and A. Sitarz for helpful discussions. B.J. and P.S. thank the LPM Université de Montpellier for hospitality and P.S. thanks the participants of the Torino conference, in particular A. Schwarz, for interesting and clarifying discussions and the organizers for a very nice conference.

References

[1] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909**, 032 (1999) [hep-th/9908142].

[2] A. Connes, M. R. Douglas and A. Schwarz, "Noncommutative geometry and matrix theory: Compactification on tori," JHEP **9802**, 003 (1998) [hep-th/9711162].

[3] J. Madore, S. Schraml, P. Schupp and J. Wess, "Gauge theory on noncommutative spaces," Eur. Phys. J. **C16**, 161 (2000) [hep-th/0001203].

[4] B. Jurco and P. Schupp, "Noncommutative Yang-Mills from equivalence of star products," Eur. Phys. J. **C14**, 367 (2000) [hep-th/0001032].

[5] B. Jurco, P. Schupp and J. Wess, "Noncommutative gauge theory for Poisson manifolds," Nucl. Phys. **B584**, 784 (2000) [hep-th/0005005].

[6] L. Bonora, M. Schnabl, M. M. Sheikh-Jabbari and A. Tomasiello, "Noncommutative SO(n) and Sp(n) gauge theories," Nucl. Phys. **B589**, 461 (2000) [hep-th/0006091].

[7] B. Jurco, S. Schraml, P. Schupp and J. Wess, "Enveloping algebra valued gauge transformations for non-Abelian gauge groups on non-commutative spaces," Eur. Phys. J. **C17**, 521 (2000) [hep-th/0006246].

[8] B. Jurco, P. Schupp and J. Wess, "Nonabelian noncommutative gauge theory," in preparation [LMU-TPW 00-20].

[9] N. Ishibashi, "A relation between commutative and noncommutative descriptions of D-branes," hep-th/9909176.

[10] K. Okuyama, "A path integral representation of the map between commutative and noncommutative gauge fields," JHEP **0003**, 016 (2000) [hep-th/9910138].

[11] L. Cornalba, "D-brane physics and noncommutative Yang-Mills theory," hep-th/9909081.

[12] L. Cornalba, "On the general structure of the non-Abelian Born-Infeld action," hep-th/0006018.

[13] T. Mehen and M. B. Wise, "Generalized *-products, Wilson lines and the solution of the Seiberg-Witten equations," hep-th/0010204.

[14] H. Liu, "*-Trek II: *n operations, open Wilson lines and the Seiberg-Witten map," hep-th/0011125.

[15] N. Seiberg, "A note on background independence in noncommutative gauge theories, matrix model and tachyon condensation," JHEP**0009**, 003 (2000) [hep-th/0008013].

[16] A. S. Cattaneo, G. Felder, "A path integral approach to the Kontsevich quantization formula," Commun. Math. Phys. **212**, 591 (2000) [math/9902090].

[17] J. Moser, "On the volume elements on a manifold," Trans. Amer. Math. Soc. **120**, 286 (1965).

[18] M. Kontsevich, "Deformation quantization of Poisson manifolds, I," q-alg/9709040.

[19] D. Manchon, "Poisson bracket, deformed bracket and gauge group actions in Kontsevich deformation quantization," math/0003004.